

# Some upper bounds for the signless Laplacian spectral radius of digraphs \*

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## Abstract

Let  $G = (V(G), E(G))$  be a digraph without loops and multiarcs, where  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $E(G)$  are the vertex set and the arc set of  $G$ , respectively. Let  $d_i^+$  be the outdegree of the vertex  $v_i$ . Let  $A(G)$  be the adjacency matrix of  $G$  and  $D(G) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$  be the diagonal matrix with outdegrees of the vertices of  $G$ . Then we call  $Q(G) = D(G) + A(G)$  the signless Laplacian matrix of  $G$ . The spectral radius of  $Q(G)$  is called the signless Laplacian spectral radius of  $G$ , denoted by  $q(G)$ . In this paper, some upper bounds for  $q(G)$  are obtained. Furthermore, some upper bounds on  $q(G)$  involving outdegrees and the average 2-outdegrees of the vertices of  $G$  are also derived.

**Key Words:** Digraph, Signless Laplacian spectral radius, Upper bounds.

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## 1 Introduction

Let  $G = (V(G), E(G))$  be a digraph without loops and multiarcs, where  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $E(G)$  are the vertex set and the arc set of  $G$ , respectively. If  $(v_i, v_j)$  be an arc of  $G$ , then  $v_i$  is called the initial vertex of this arc and  $v_j$  is called the terminal vertex of this arc. For any vertex  $v_i$  of  $G$ , we denote  $N_i^+ = N_{v_i}^+(G) = \{v_j : (v_i, v_j) \in E(G)\}$  and  $N_i^- = N_{v_i}^-(G) = \{v_j : (v_j, v_i) \in E(G)\}$  the set of out-neighbors and in-neighbors of  $v_i$ , respectively. Let  $d_i^+ = |N_i^+|$  denote the outdegree of the vertex  $v_i$  and  $d_i^- = |N_i^-|$  denote

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the indegree of the vertex  $v_i$  in the digraph  $G$ . The maximum vertex outdegree is denoted by  $\Delta^+$ , and the minimum outdegree by  $\delta^+$ . If  $\delta^+ = \Delta^+$ , then  $G$  is a regular digraph. Let  $t_i^+ = \sum_{v_j \in N_i^+} d_j^+$  be the 2-outdegree of the vertex  $v_i$ ,  $m_i^+ = \frac{t_i^+}{d_i^+}$  the average 2-outdegree of the vertex  $v_i$ . A digraph is strongly connected if for every pair of vertices  $v_i, v_j \in V(G)$ , there exists a directed path from  $v_i$  to  $v_j$  and a directed path from  $v_j$  to  $v_i$ . In this paper, we consider finite digraphs without loops and multiarcs, which have at least one arc.

For a digraph  $G$ , let  $A(G) = (a_{ij})$  denote the adjacency matrix of  $G$ , where  $a_{ij} = 1$  if  $(v_i, v_j) \in E(G)$  and  $a_{ij} = 0$  otherwise. Let  $D(G) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$  be the diagonal matrix with outdegrees of the vertices of  $G$  and  $Q(G) = D(G) + A(G)$  the signless Laplacian matrix of  $G$ . However, the signless Laplacian matrix of an undirected graph  $D$  can be treated as the signless Laplacian matrix of the digraph  $G'$ , where  $G'$  is obtained from  $D$  by replace each edge with pair of oppositely directed arcs joining the same pair of vertices. Therefore, the research of the signless Laplacian matrix of a digraph has more universal significance than undirected graph.

The eigenvalues of  $Q(G)$  are called the signless Laplacian eigenvalues of  $G$ , denoted by  $q_1, q_2, \dots, q_n$ . In general  $Q(G)$  are not symmetric and so its eigenvalues can be complex numbers. We usually assume that  $|q_1| \geq |q_2| \geq \dots \geq |q_n|$ . The signless Laplacian spectral radius of  $G$  is denoted and defined as  $q(G) = |q_1|$ , i.e., the largest absolute value of the signless Laplacian eigenvalues of  $G$ . Since  $Q(G)$  is a nonnegative matrix, it follows from Perron Frobenius Theorem that  $q(G) = q_1$  is a real number.

For the Laplacian spectral radius and signless Laplacian spectral radius of an undirected graph are well treated in the literature, see [12, 13, 14, 16] and [3, 4, 6, 7, 8, 14], respectively. Recently, there are some papers that give some lower or upper bounds for the spectral radius of a digraph, see [2, 5, 15]. Now we consider the signless Laplacian spectral radius of a digraph  $G$ . For application it is crucial to be able to computer or at least estimate  $q(G)$  for a given digraph.

In 2013, S.B. Bozkurt and D. Bozkurt in [1] obtained the following bounds for signless Laplacian spectral radius of a digraph.

$$q(G) \leq \max\{d_i^+ + d_j^+ : (v_i, v_j) \in E(G)\}. \quad (1)$$

$$q(G) \leq \max\{d_i^+ + m_i^+ : v_i \in V(G)\}. \quad (2)$$

$$q(G) \leq \max \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} : (v_i, v_j) \in E(G) \right\}. \quad (3)$$

$$q(G) \leq \max \left\{ d_i^+ + \sqrt{\sum_{v_j: (v_j, v_i) \in E(G)} d_j^+} : v_i \in V(G) \right\}. \quad (4)$$

In 2014, Hong and You in [9] gave a sharp bound for the signless Laplacian spectral radius of a digraph:

$$q(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8 \sum_{k=1}^{i-1} (d_k^+ - d_i^+)}}{2} \right\}. \quad (5)$$

**Remark 1.1.** *Note that  $G$  is a strongly connected digraph for bounds (1), (3), (4), respectively.*

In this paper, we study on the signless Laplacian spectral radius of a digraph  $G$ . We obtain some upper bounds for  $q(G)$ , and we also show that some upper bounds on  $q(G)$  involving outdegrees and the average 2-outdegrees of the vertices of  $G$  can be obtained from our bounds.

## 2 Preliminaries Lemmas

In this section, we give the following lemmas which will be used in the following study.

**Lemma 2.1.** ([10]) *Let  $M = (m_{ij})$  be an  $n \times n$  nonnegative matrix with spectral radius  $\rho(M)$ , i.e., the largest eigenvalues of  $M$ , and let  $R_i = R_i(M)$  be the  $i$ -th row sum of  $M$ , i.e.,  $R_i(M) = \sum_{j=1}^n m_{ij}$  ( $1 \leq i \leq n$ ). Then*

$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M) : 1 \leq i \leq n\}. \quad (6)$$

*Moreover, if  $M$  is irreducible, then any equality holds in (6) if and only if  $R_1 = R_2 = \dots = R_n$ .*

**Lemma 2.2.** ([10]) *Let  $M$  be an irreducible nonnegative matrix. Then  $\rho(M)$  is an eigenvalue of  $M$  and there is a positive vector  $X$  such that  $MX = \rho(M)X$ .*

**Lemma 2.3.** ([11]) *Let  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ ,  $r_i = \sum_{j \neq i} |a_{ij}|$  for each  $i = 1, 2, \dots, n$ ,  $S_{ij} = \{z \in \mathbb{C} : |z - a_{ii}| \cdot |z - a_{jj}| \leq r_i r_j\}$  for all  $i \neq j$ . Also let  $E(A) = \{(i, j) : a_{ij} \neq 0, 1 \leq i \neq j \leq n\}$ . If  $A$  is irreducible, then all eigenvalues of  $A$  are contained in the following region*

$$\Omega(A) = \bigcup_{(i,j) \in E(A)} S_{ij}. \quad (7)$$

*Furthermore, a boundary point  $\lambda$  of (7) can be an eigenvalue of  $A$  only if  $\lambda$  locates on the boundary of each oval region  $S_{ij}$  for  $e_{ij} \in E(A)$ .*

## 3 Some upper bounds for the signless Laplacian spectral radius of digraphs

In this section, we present some upper bounds for the signless Laplacian spectral radius  $q(G)$  of a digraph  $G$  and also show that some bounds involving outdegrees, the average 2-outdegrees, the maximum outdegree and the minimum outdegree of the vertices of  $G$  with  $n$  vertices and  $m$  arcs can be obtained from our bounds.

**Theorem 3.1.** *Let  $G$  be a strongly connected digraph with  $n \geq 3$  vertices,  $m$  arcs, the maximum vertex outdegree  $\Delta^+$  and the minimum outdegree  $\delta^+$ . Then*

$$q(G) \leq \max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\}. \quad (8)$$

Moreover, if  $G(\neq \overrightarrow{C_n})$  is a regular digraph or  $G \cong \overleftrightarrow{K_{1,n-1}}$ , where  $\overleftrightarrow{K_{1,n-1}}$  denotes the digraph on  $n$  vertices which replace each edge in star graph  $K_{1,n-1}$  with the pair of oppositely directed arcs, then the equality holds in (8)

*Proof.* From (2), we know that  $q(G) \leq \max\{d_i^+ + m_i^+ : v_i \in V(G)\}$ . So we only need to prove that  $\max\{d_i^+ + m_i^+ : v_i \in V(G)\} \leq \max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\}$ . Suppose  $\max\{d_i^+ + m_i^+ : v_i \in V(G)\}$  occurs at vertex  $u$ . Two cases arise  $d_u^+ = 1$ , or  $2 \leq d_u^+ \leq \Delta^+$ .

**Case 1.**  $d_u^+ = 1$ . Suppose that  $N_u^+ = \{w\}$ . Since  $m_u^+ = d_w^+ \leq \Delta^+$ , thus  $d_u^+ + m_u^+ \leq 1 + \Delta^+$ . Since  $\sum_{v_i \in V(G)} d_i^+ = m$ , let  $d_j^+ = \Delta^+$ , then  $\sum_{i \neq j} d_i^+ = m - \Delta^+ \geq (n-1)\delta^+$ , so

$m - (n-1)\delta^+ \geq \Delta^+$ . Therefore  $\delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+} \geq \delta^+ - 1 + \frac{\Delta^+}{\Delta^+} = \delta^+ \geq 1$ . Thus  $d_u^+ + m_u^+ \leq 1 + \Delta^+ \leq \Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}$ , the result follows.

**Case 2.**  $2 \leq d_u^+ \leq \Delta^+$ . Note that  $m - (n-1)\delta^+ \geq d_u^+ \geq 2$ , and

$$\begin{aligned} m &= \sum_{v:(u,v) \in E(G)} d_v^+ + \sum_{v:(u,v) \notin E(G)} d_v^+ \\ &\geq \sum_{v:(u,v) \in E(G)} d_v^+ + d_u^+ + (n - d_u^+ - 1)\delta^+, \end{aligned}$$

thus

$$\begin{aligned} \sum_{v:(u,v) \in E(G)} d_v^+ &\leq m - d_u^+ - (n - d_u^+ - 1)\delta^+ \\ &= m - (n-1)\delta^+ + (\delta^+ - 1)d_u^+ \\ m_u^+ &= \frac{\sum_{v:(u,v) \in E(G)} d_v^+}{d_u^+} \leq \frac{m - (n-1)\delta^+}{d_u^+} + \delta^+ - 1. \end{aligned}$$

This follows that  $m_u^+ + d_u^+ \leq d_u^+ + \frac{m - (n-1)\delta^+}{d_u^+} + \delta^+ - 1$ . Let  $f(x) = x + \frac{m - (n-1)\delta^+}{x} + \delta^+ - 1$ , where  $x \in [2, \Delta^+]$ . It is easy to see that  $f'(x) = 1 - \frac{m - (n-1)\delta^+}{x^2}$ . Let  $a = m - (n-1)\delta^+$ , then  $\sqrt{a}$  is the unique positive root of  $f'(x) = 0$ . We consider the next three Subcases.

**Subcase 1.**  $\sqrt{a} < 2$ . When  $x \in [2, \Delta^+]$ , since  $f'(x) > 0$ , then  $f(x) \leq f(\Delta^+)$ .

**Subcase 2.**  $2 \leq \sqrt{a} \leq \Delta^+$ . Then  $f'(x) < 0$  for  $x \in [2, \sqrt{a}]$ , and  $f'(x) \geq 0$ , for  $x \in [\sqrt{a}, \Delta^+]$ . Thus,  $f(x) \leq \max\{f(2), f(\Delta^+)\}$ .

**Subcase 3.**  $\Delta^+ < \sqrt{a}$ . When  $x \in [2, \Delta^+]$ , since  $f'(x) < 0$ , then  $f(x) \leq f(2)$ .

Recall that  $2 \leq d_u^+ \leq \Delta^+$ , thus

$$m_u^+ + d_u^+ \leq \max\{f(2), f(\Delta^+)\}$$

$$= \max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\}.$$

If  $G(\neq \overrightarrow{C_n})$  is a regular digraph, then  $d_i^+ + m_i^+ = 2d_i^+ = 2\Delta^+$  for all  $v_i \in V(G)$ . We can get  $q(G) = 2\Delta^+$ . Since  $G(\neq \overrightarrow{C_n})$  is a strongly connected digraph, then we may assume that  $\Delta^+ \geq 2$ , this implies that  $\delta^+ + 1 + \frac{m - \delta^+(n-1)}{2} = \Delta^+ + 1 + \frac{\Delta^+}{2} \leq 2\Delta^+ = \Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}$ . So  $\max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\} = 2\Delta^+$ . Thus, the equality holds. If  $G \cong \overleftrightarrow{K}_{1,n-1}$ , we can get  $q(G) = n$ . Since  $\delta^+ + 1 + \frac{m - \delta^+(n-1)}{2} = 2 + \frac{n-1}{2} \leq n$  from  $n \geq 3$  and  $\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+} = n - 1 + 1 - 1 + \frac{n-1}{n-1} = n$ . So  $\max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\} = n$ . Thus, the equality also holds. By combining the above discussion, the result follows.  $\square$

**Corollary 3.2.** *Let  $G$  be a strongly connected digraph with  $n \geq 3$  vertices,  $m$  arcs, the maximum outdegree  $\Delta^+$  and the minimum outdegree  $\delta^+$ . If  $\Delta^+ \geq \frac{m-(n-1)}{2}$  and  $\delta^+ = 1$ , then*

$$q(G) \leq \Delta^+ + 2. \quad (9)$$

*Proof.* Because  $\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+} \leq \Delta^+ + 2$ ,  $\delta^+ + 1 + \frac{m - \delta^+(n-1)}{2} \leq \Delta^+ + 2$ , therefore by Theorem 3.1, we have  $q(G) \leq \Delta^+ + 2$ .  $\square$

Let  $G^*(m, n, \frac{m-(n-1)}{2}, 1)$  be a class of strongly connected digraphs with  $\Delta^+ \geq \frac{m-(n-1)}{2}$ ,  $\delta^+ = 1$ , and there exists a vertex  $v_0 \in V(G)$  such that  $d_{v_0} = \Delta^+$  and there exists a vertex  $v_k \in N_{v_0}^+$ ,  $d_{v_k}^+ \geq 2$ .

**Remark 3.3.** *For  $G \in G^*(m, n, \frac{m-(n-1)}{2}, 1)$ , we have  $\Delta^+ + 2 \leq \max\{d_i^+ + d_j^+ : (v_i, v_j) \in E(G)\}$ , thus the upper bound (9) is better than the upper bound (1) for the class of digraphs  $G \in G^*(m, n, \frac{m-(n-1)}{2}, 1)$ . But for general digraphs, the upper bound (9) is incomparable with the upper bound (1).*

**Example 3.4.** *Let  $G$  be the digraph of order 4, as shown in Figure 1. Since it has 9 arcs, and the maximum outdegree  $\Delta^+ = 3 = \frac{9-(4-1)}{2}$ , the minimum outdegree  $\delta^+ = 1$ , and there exists a vertex  $v_4 \in N_{v_1}^+$ ,  $d_{v_4}^+ = 3 > 2$ , therefore  $G = G^*(9, 4, 3, 1)$ .*

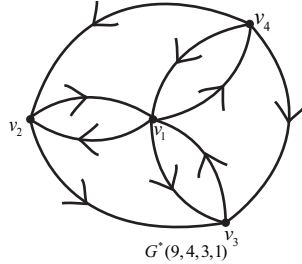


Figure 1: Graph  $G^*(9, 4, 3, 1)$

Table 1: Values of the upper bounds for example 1.

| $q(G)$            | (1)    | (9) |
|-------------------|--------|-----|
| $G^*(9, 4, 3, 1)$ | 4.7321 | 6   |
|                   |        | 5   |

**Theorem 3.5.** *Let  $G$  be a strongly connected digraph with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and arc set  $E(G)$ . Then*

$$q(G) \leq \max \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4\sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}}}{2} : (v_i, v_j) \in E(G) \right\}. \quad (10)$$

Moreover if  $G$  is a regular digraph or a bipartite semiregular digraph, then the equality holds in (10).

*Proof.* From the definition of  $D = D(G)$  we get  $D^{\frac{1}{2}} = \text{diag}(\sqrt{d_i^+} : v_i \in V(G))$ , and consider the similar matrix  $P = D^{-\frac{1}{2}}Q(G)D^{\frac{1}{2}}$ . Since  $G$  is a strongly connected digraph, it is easy to see that  $P$  is irreducible and nonnegative. Now the  $(i, j)$ -th element of  $P = D^{-\frac{1}{2}}Q(G)D^{\frac{1}{2}}$  is

$$p_{ij} = \begin{cases} d_i^+ & \text{if } i = j, \\ \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} & \text{if } (v_i, v_j) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let  $R_i(P)$  be the  $i$ -th row sum of  $P$  and  $R'_i(P) = R_i - d_i^+$ . Then by Cauchy-Schwarz inequality, we have

$$\begin{aligned} R'_i(P)^2 &= \left( \sum_{v_j: (v_i, v_j) \in E(G)} \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} \right)^2 \leq \sum_{v_j: (v_i, v_j) \in E(G)} 1^2 \sum_{v_j: (v_i, v_j) \in E(G)} \frac{d_j^+}{d_i^+} \\ &= \sum_{v_j: (v_i, v_j) \in E(G)} d_j^+ = d_i^+ m_i^+. \end{aligned}$$

Since  $P$  is irreducible and nonnegative,  $\rho(P)$  denotes the spectral radius of  $P$ . Then by Lemma 2.3, there at least exists  $(v_i, v_j) \in E(G)$  such that  $\rho(P)$  is contained in the following oval region

$$|\rho(P) - d_i^+| |\rho(P) - d_j^+| \leq R'_i(P) R'_j(P) \leq \sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}.$$

Obviously,  $\rho(P) = q(G) > \max\{d_i^+ : v_i \in E(G)\}$ , and  $(\rho(P) - d_i^+)(\rho(P) - d_j^+) \leq |\rho(P) - d_i^+| |\rho(P) - d_j^+|$ . Therefore, solving the above inequality we obtain

$$q(G) \leq \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4\sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}}}{2} \right\}.$$

Hence (10) holds.

If  $G$  is a regular digraph,

$$q(G) = 2\Delta^+ = \max \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4\sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}}}{2} : (v_i, v_j) \in E(G) \right\}.$$

Thus, the equality holds.

If  $G$  is a bipartite semiregular digraph,

$$\max \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4\sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}}}{2} : (v_i, v_j) \in E(G) \right\} = d_i^+ + d_j^+.$$

Because  $q(G) = \rho(D^{-1}Q(G)D)$ , the  $i$ -th row sum of  $D^{-1}Q(G)D$  is  $d_i^+ + m_i^+$ , and  $G$  is a bipartite semiregular digraph, therefore  $d_i^+ + m_i^+ = d_i^+ + d_j^+, (v_i, v_j) \in E(G)$ , that is the row sums of  $D^{-1}Q(G)D$  are all equal, then by Lemma 2.1,  $\rho(D^{-1}Q(G)D) = d_i^+ + d_j^+$ . Thus we have

$$\begin{aligned} q(G) &= \rho(D^{-1}Q(G)D) = d_i^+ + d_j^+ \\ &= \max \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4\sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}}}{2} : (v_i, v_j) \in E(G) \right\}. \end{aligned}$$

Then the equality holds.  $\square$

For a digraph  $G = (V(G), E(G))$ , let  $f : V(G) \times V(G) \rightarrow \mathbb{R}$  be a function. If  $f(v_i, v_j) > 0$  for all  $(v_i, v_j) \in E(G)$ , we say  $f$  is positive on arcs.

**Theorem 3.6.** *Let  $G = (V(G), E(G))$  be a digraph. Let  $f : V(G) \times V(G) \rightarrow \mathbb{R}^+ \cup \{0\}$  be a nonnegative function which is positive on arcs. Then*

$$q(G) \leq \max \left\{ \frac{\sum_{v_k: (v_i, v_k) \in E(G)} f(v_i, v_k) + \sum_{v_k: (v_j, v_k) \in E(G)} f(v_j, v_k)}{f(v_i, v_j)} : (v_i, v_j) \in E(G) \right\}. \quad (11)$$

*Proof.* Let  $\mathbf{X} = (x_1, x_2, \dots, x_n)^T$  be an eigenvector corresponding to the eigenvalue  $q(G)$  of  $Q(G)$ . Since

$$Q(G)\mathbf{X} = q(G)\mathbf{X}.$$

Then for  $1 \leq i \leq n$

$$q(G)x_i = d_i^+ x_i + \sum_{v_k: (v_i, v_k) \in E(G)} x_k = \sum_{v_k: (v_i, v_k) \in E(G)} (x_i + x_k). \quad (12)$$

By (12), we have

$$q(G)(x_i + x_j) = \sum_{v_k: (v_i, v_k) \in E(G)} (x_i + x_k) + \sum_{v_k: (v_j, v_k) \in E(G)} (x_j + x_k).$$

For convenience we use  $f(i, j)$  denote  $f(v_i, v_j)$ . Set  $g(i, j) = \frac{x_i + x_j}{f(i, j)}$ . If  $(v_i, v_j) \in E(G)$ , then

$$q(G)f(i, j)g(i, j) = \sum_{v_k: (v_i, v_k) \in E(G)} f(i, k)g(i, k) + \sum_{v_k: (v_j, v_k) \in E(G)} f(j, k)g(j, k). \quad (13)$$

By (13), we get

$$\begin{aligned} |q(G)f(i, j)g(i, j)| &= q(G)f(i, j)|g(i, j)| \\ &\leq \sum_{v_k: (v_i, v_k) \in E(G)} f(i, k)|g(i, k)| + \sum_{v_k: (v_j, v_k) \in E(G)} f(j, k)|g(j, k)|. \end{aligned}$$

Now choose  $i_1, j_1$  such that  $(v_{i_1}, v_{j_1}) \in E(G)$  and  $|g(i_1, j_1)| = \max\{|g(i, j)| : (v_i, v_j) \in E(G)\}$ . If  $|g(i_1, j_1)| = 0$ , then  $|g(i, j)| = 0$  for all arcs  $(v_i, v_j) \in E(G)$ . i.e.,  $x_i + x_j = 0$  for all arcs  $(v_i, v_j) \in E(G)$ . By (12), we have  $q(G) = 0$  which is impossible, since  $G$  has at least one arc. So  $|g(i_1, j_1)| > 0$ . Then

$$q(G)f(i_1, j_1)|g(i_1, j_1)| \leq \sum_{v_k: (v_{i_1}, v_k) \in E(G)} f(i_1, k)|g(i_1, k)| + \sum_{v_k: (v_{j_1}, v_k) \in E(G)} f(j_1, k)|g(j_1, k)|.$$

Therefore, we obtain

$$\begin{aligned} q(G) &\leq \sum_{v_k: (v_{i_1}, v_k) \in E(G)} \frac{f(i_1, k)}{f(i_1, j_1)} \frac{|g(i_1, k)|}{|g(i_1, j_1)|} + \sum_{v_k: (v_{j_1}, v_k) \in E(G)} \frac{f(j_1, k)}{f(i_1, j_1)} \frac{|g(j_1, k)|}{|g(i_1, j_1)|} \\ &\leq \sum_{v_k: (v_{i_1}, v_k) \in E(G)} \frac{f(i_1, k)}{f(i_1, j_1)} + \sum_{v_k: (v_{j_1}, v_k) \in E(G)} \frac{f(j_1, k)}{f(i_1, j_1)}, \end{aligned}$$

i.e.,

$$q(G) \leq \frac{\sum_{v_k: (v_{i_1}, v_k) \in E(G)} f(i_1, k) + \sum_{v_k: (v_{j_1}, v_k) \in E(G)} f(j_1, k)}{f(i_1, j_1)}, \text{ where } (v_{i_1}, v_{j_1}) \in E(G).$$

This proves the desired result.  $\square$

**Corollary 3.7.** *Let  $G = (V(G), E(G))$  be a digraph. Then*

$$q(G) \leq \max \left\{ d_i^+ \sqrt{\frac{m_i^+}{d_j^+}} + d_j^+ \sqrt{\frac{m_j^+}{d_i^+}} : (v_i, v_j) \in E(G) \right\}. \quad (14)$$

*Proof.* Setting  $f(v_i, v_j) = \sqrt{d_i^+ d_j^+}$  in (11), by Cauchy-Schwarz inequality,

$$\sum_{v_k: (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k: (v_i, v_k) \in E(G)} \sqrt{d_i^+ d_k^+} = \sum_{v_k: (v_i, v_k) \in E(G)} (\sqrt{d_i^+} \sqrt{d_k^+})$$



$$\leq \sqrt{\sum_{v_k:(v_i,v_k) \in E(G)} d_i^+ \sum_{v_k:(v_i,v_k) \in E(G)} d_k^+} = \sqrt{d_i^{+2} \sum_{v_k:(v_i,v_k) \in E(G)} d_k^+} = d_i^+ \sqrt{d_i^+ m_i^+}.$$

By (11), we get

$$\begin{aligned} q(G) &\leq \max \left\{ \frac{\sum_{v_k:(v_i,v_k) \in E(G)} f(v_i, v_k) + \sum_{v_k:(v_j,v_k) \in E(G)} f(v_j, v_k)}{f(v_i, v_j)} : (v_i, v_j) \in E(G) \right\} \\ &\leq \max \left\{ \frac{d_i^+ \sqrt{d_i^+ m_i^+} + d_j^+ \sqrt{d_j^+ m_j^+}}{\sqrt{d_i^+ d_j^+}} : (v_i, v_j) \in E(G) \right\} \\ &= \max \left\{ d_i^+ \sqrt{\frac{m_i^+}{d_j^+}} + d_j^+ \sqrt{\frac{m_j^+}{d_i^+}} : (v_i, v_j) \in E(G) \right\}. \end{aligned}$$

□

**Corollary 3.8.** *Let  $G = (V(G), E(G))$  be a digraph. Then*

$$q(G) \leq \max \left\{ \frac{d_i^+(d_i^+ + m_i^+) + d_j^+(d_j^+ + m_j^+)}{d_i^+ + d_j^+} : (v_i, v_j) \in E(G) \right\}. \quad (15)$$

*Proof.* Setting  $f(v_i, v_j) = d_i^+ + d_j^+$  in (11), since  $\sum_{v_k:(v_i,v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k:(v_i,v_k) \in E(G)} (d_i^+ + d_k^+) = d_i^+(d_i^+ + m_i^+)$ , So we get the desired result. □

**Corollary 3.9.** *Let  $G = (V(G), E(G))$  be a digraph. Then*

$$q(G) \leq \max \left\{ \frac{d_i^+ \sqrt{d_i^+ + m_i^+} + d_j^+ \sqrt{d_j^+ + m_j^+}}{\sqrt{d_i^+ + d_j^+}} : (v_i, v_j) \in E(G) \right\}. \quad (16)$$

*Proof.* Setting  $f(v_i, v_j) = \sqrt{d_i^+ + d_j^+}$  in (11), since

$$\begin{aligned} \sum_{v_k:(v_i,v_k) \in E(G)} f(v_i, v_k) &= \sum_{v_k:(v_i,v_k) \in E(G)} (1 \cdot \sqrt{d_i^+ + d_k^+}) \leq \sqrt{d_i^+} \sum_{v_k:(v_i,v_k) \in E(G)} (d_i^+ + d_k^+) \\ &= \sqrt{d_i^+(d_i^{+2} + d_i^+ m_i^+)} = d_i^+ \sqrt{d_i^+ + m_i^+} \text{ by Cauchy-Schwarz inequality.} \end{aligned}$$

Thus by (11) we get the desired result. □

**Corollary 3.10.** *Let  $G = (V(G), E(G))$  be a digraph. Then*

$$q(G) \leq \max \left\{ \frac{d_i^+(\sqrt{d_i^+} + \sqrt{m_i^+}) + d_j^+(\sqrt{d_j^+} + \sqrt{m_j^+})}{\sqrt{d_i^+} + \sqrt{d_j^+}} : (v_i, v_j) \in E(G) \right\}. \quad (17)$$

*Proof.* Setting  $f(v_i, v_j) = \sqrt{d_i^+} + \sqrt{d_j^+}$  in (11), since  $\sum_{v_k: (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k: (v_i, v_k) \in E(G)} (\sqrt{d_i^+} + \sqrt{d_k^+}) = d_i^+ \sqrt{d_i^+} + \sum_{v_k: (v_i, v_k) \in E(G)} (1 \cdot \sqrt{d_k^+}) \leq d_i^{+\frac{3}{2}} + \sqrt{d_i^+ \sum_{v_k: (v_i, v_k) \in E(G)} d_k^+} = d_i^+ (\sqrt{d_i^+} + \sqrt{m_i^+})$  by Cauchy-Schwarz inequality. By (11) the result follows.  $\square$

Notice that (16) and (17) can be viewed as adding square roots to (15) at difference places.

## 4 Example

Let  $G_1, G_2$  be the digraphs of order 4,6, respectively, as shown in Figure 2.

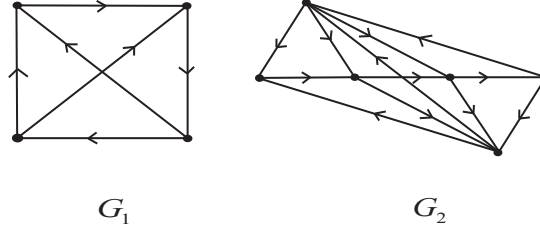


Figure 2:

Table 2: Values of the various bounds for example 1.

|       | $q(G)$ | (1)    | (2)    | (3)    | (4)    | (5)    |
|-------|--------|--------|--------|--------|--------|--------|
|       | (8)    | (10)   | (14)   | (15)   | (16)   | (17)   |
| $G_1$ | 3.0000 | 4.0000 | 3.5000 | 3.3028 | 3.4142 | 3.5616 |
|       | 3.5000 | 3.5651 | 3.4495 | 3.3333 | 3.6029 | 3.5731 |
| $G_2$ | 4.1984 | 5.0000 | 4.6667 | 4.6016 | 5.0000 | 4.7321 |
|       | 5.5000 | 4.7913 | 4.5644 | 4.6000 | 4.7956 | 4.7866 |

**Remark 4.1.** Obviously, from Table 1, the bound (3) is the best in all known upper bounds for  $G_1$ , and the bound (14) is the best for  $G_2$ . Finally bound (15) is the second-best bounds for  $G_1$  and  $G_2$ . In general, these bounds are incomparable.

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